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PROOF FOR A CASE WHERE DISCOUNTING ADVANCES THE DOOMSDAY

by Tjalling C. Koopmans*

In a previous paper (Koopmans [1973]), I considered some problems of "optimal" consumption $\hat{r}_t$ over time of an exhaustible resource of known finite total availability R. In one of the cases studied, consumption of a minimum amount of the resource is assumed to be essential to human life, in such a way that all life ceases upon its exhaustion at time T. Assuming a constant population until that time, and denoting by $\underline{r}$ the positive minimum consumption level needed for survival of that population, the survival period T is constrained by

$$(1) \qquad 0 < T \leqq R/\underline{r} \equiv \overline{T} \quad .$$

Here equality ($T=\overline{T}$) can be attained only by consuming at the minimum level ($r_t=\underline{r}$) at all times, $0 \leqq t \leqq \overline{T}$.

However, optimality is defined in terms of maximization of the integral over time of discounted future utility levels,

$$(2) \qquad V(\rho,T,(r_t)) \equiv \int_0^T e^{-\rho t} v(r_t)dt \quad ,$$

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where ρ is a discount rate, $\rho \geqq 0$, applied in continuous time to the utility flow $v(r_t)$ arising at any time t from a consumption flow r_t of the resource. The utility flow function $v(r)$ is defined for $r \geqq \underline{r}$, is twice continuously differentiable and satisfies

$$\text{(3a,b,c,d)} \quad v'(r) > 0,\ v''(r) < 0 \quad \text{for } r > \underline{r},\ v(\underline{r}) = 0,$$
$$\lim_{r \to \underline{r}} v'(r) = \infty .$$

That is, $v(r)$ is (a) strictly increasing and (b) strictly concave. The stipulation (c) anchors the utility scale. Some such anchoring, though not necessarily the given one, is needed whenever population size is a decision variable. The last requirement (d) simplifies a step in the proof, and can be secured if needed by a distortion of $v(r)$ in a neighborhood of $\underline{r}$ that does not affect the solution.

The paper referred to gives an intuitive argument for the following

<u>Theorem</u>: <u>For each</u> $\rho \geqq 0$ <u>there exists a unique optimal path</u> $r_t = \hat{r}_t$, $0 \leqq t \leqq \hat{T}_\rho$, <u>maximizing</u> (2) <u>subject to</u>

$$(4)\begin{cases} \text{(4a)} \quad r_t \text{ is a continuous function on } [0,T] , \\ \text{(4b)} \quad \int_0^T r_t dt \leqq R,\ r_t \geqq \underline{r},\ 0 \leqq t \leqq T . \end{cases}$$

<u>For</u> $\rho = 0$, <u>the optimal path</u> $(\hat{r}_t | 0 \leqq t \leqq \hat{T}_0)$ <u>is defined by</u>

$$(5)\begin{cases} (5a) & \hat{r}_t = \hat{r}, \text{ a constant, for } 0 \leqq t \leqq \hat{T}_0 \ , \\ (5b) & v(\hat{r}) = \hat{r}v'(\hat{r}) \ , \\ (5c) & \hat{r}\hat{T}_0 = R \ . \end{cases}$$

<u>For</u> $\rho > 0$ <u>it is defined by</u>

$$(6)\begin{cases} (6a) & e^{-\rho t}v'(\hat{r}_t) = e^{-\rho\hat{T}_\rho}v'(\hat{r}) \ , \quad 0 \leqq t \leqq \hat{T}_\rho \ , \quad \hat{r} \text{ as in (5b)} \ , \\ (6b) & \int_0^{\hat{T}_\rho} \hat{r}_t dt = R \ . \end{cases}$$

The diagram illustrates the solution. For $\rho = 0$, (6) implies (5), and consumption of the resource is constant during survival. Its optimal level $\hat{r}$ is obtained in (5b,c) by balancing the number of years of survival against the constant level of utility flow that the total resource stock makes possible during survival. Since $\hat{r} > \underline{r}$, the optimum survival period $\hat{T}_0$ is shorter than the maximum $\overline{T}$ defined by (1).

For $\rho > 0$, the optimal path $\hat{r}_t$ follows a declining curve given by (6a), which starts from a level $\hat{r}_0$ such that, when resource exhaustion brings life to a stop at time $t = \hat{T}_\rho$, the level $\hat{r}_{\hat{T}_\rho} = \hat{r}$ is just reached. Since the decline is steeper when ρ is larger, the survival period is shorter, the larger is ρ - which explains the title of this note.

The intuitive argument already referred to gives insight into the theorem; the following proof establishes its validity.

Proof: We first consider paths optimal under the added constraint of some arbitrarily fixed value $T = T^*$ of T satisfying $0 < T^* < \overline{T}$. Assume that such a " T^* - optimal" path r_t^* exists and that

$$(7) \qquad r_t^* \geqq \underline{r} + \delta \quad \text{for } 0 \leqq t \leqq T^* \text{ and some } \delta > 0 \quad .$$

Then, if s_t is a continuous function defined for $0 \leqq t \leqq T^*$ such that

$$(8) \qquad |s_t| \leqq \delta \quad , \qquad \int_0^{T^*} s_t dt = 0 \quad ,$$

the path

$$(9) \qquad r_t = r_t^* + \varepsilon s_t \quad , \qquad 0 \leqq t \leqq T^* \quad ,$$

is T^*-feasible for $|\varepsilon| \leqq 1$ and satisfies

$$(10) \begin{cases} (10a) \begin{cases} V(\rho, T^*, (r_t)) - V(\rho, T^*, (r_t^*)) = \\ = \int_0^{T^*} e^{-\rho t}(v(r_t) - v(r_t^*))dt = \end{cases} \\ (10b) \begin{cases} = \varepsilon \int_0^{T^*} e^{-\rho t} v'(r_t^*) s_t dt + R(\varepsilon) \quad , \end{cases} \end{cases}$$

where the remainder $R(\varepsilon)$ is of second order in ε. It is therefore a necessary condition for the T^*-optimality of r_t^* that

$$(11) \qquad p_t \equiv e^{-\rho t} v'(r_t^*) = \text{constant} = e^{-\rho T^*} v'(r_{T^*}^*) \quad , \text{ say,}$$

because, if we had $p_{t'} \neq p_{t''}$, $0 \leqq t', t'' \leqq T^*$, we could by choosing s_t of one sign in a neighborhood in $[0,T^*]$ of t', s_t of the opposite sign in one of t'' and zero elsewhere while preserving (8) make the last member of (10) positive for some ε with $|\varepsilon| \leqq 1$.

In the light of (3a,b), (11) justifies our assumption that r_t^* is a continuous function of t. We now find that r_t^* is constant for $\rho = 0$, strictly decreasing for $\rho > 0$. Given $r_{T^*}^*$, say, the solution r_t^* of (11) is uniquely determined, and, for each t, r_t^* is a strictly increasing differentiable function of the given $r_{T^*}^*$. Also, by (3d),

$$\lim_{r_{T^*}^* \to \underline{r}} \int_0^T r_t^* dt = \int_0^T \underline{r} dt = T^* \underline{r} < \overline{T} \underline{r} = R \quad ,$$

whereas, for sufficiently large $r_{T^*}^*$,

$$\int_0^{T^*} r_t^* dt > R \quad .$$

Therefore there is a unique number $\alpha^* > \underline{r}$ such that the unique solution r_t^* of (11) with $r_{T^*}^* = \alpha^*$ satisfies

(12) $$\int_0^{T^*} r_t^* dt = R \quad .$$

From here on r_t^* will denote that path for the chosen T^*. Note that this path also satisfies (7).

To prove the unique T^*-optimality of r_t^*, let r_t be any T^*-feasible path such that $r_{t_0} \neq r_{t_0}^*$ for some $t_0 \varepsilon [0,T]$. Then, by the continuity of r_t, r_t^*, $r_t \neq r_t^*$ for all t in some neighborhood τ of t_0 in $[0,T^*]$. By (3b), for all $t\varepsilon[0,T^*]$,

(13) $$v(r_t) - v(r_t^*) \begin{bmatrix} < \\ \leqq \end{bmatrix} (r_t - r_t^*)\, v'(r_t^*) \quad \text{for} \quad t\varepsilon \begin{bmatrix} \tau \\ \tau^* \end{bmatrix} \quad ,$$

where $\tau^* \equiv [0,T^*] - \tau$. Therefore, we have from (10a), (11), (4b) with $T = T^*$, and (12) that

$$V(\rho,T^*,(r_t)) - V(\rho,T^*,(r_t^*)) =$$

$$= \left(\int_\tau + \int_{\tau^*}\right) e^{-\rho t}(v(r_t) - v(r_t^*))dt <$$

$$< \int_0^{T^*} (r_t - r_t^*) e^{-\rho t} v'(r_t^*) dt =$$

$$= e^{-\rho T^*} v'(r_{T^*}^*) \int_0^{T^*} (r_t - r_t^*) dt \leqq 0 \quad .$$

Hence r_t^* is uniquely T^*-optimal.

We now make T^* a variable, writing T instead of T^* and r_t^T instead of r_t^*. Note that, for each t, $0 \leqq t < \overline{T}$, r_t^T is a differentiable function of T for $t \leqq T < \overline{T}$. Therefore

$$V_T \equiv V(\rho,T,(r_t^T)) = \int_0^T e^{-\rho t} v(r_t^T)dt$$

is a differentiable function of T for $0 \leqq T < \overline{T}$, and

$$\frac{dV_T}{dT} = e^{-\rho T} v(r_T^T) + \int_0^T e^{-\rho t} v'(r_t^T) \frac{dr_t^T}{dT} dt =$$

$$= e^{-\rho T} v(r_T^T) + e^{-\rho T} v'(r_T^T) \int_0^T \frac{dr_t^T}{dT} dt$$

by (11). But, by (12),

$$0 = \frac{dR}{dT} = r_T^T + \int_0^T \frac{dr_t^T}{dT} dt \quad .$$

Therefore,

$$e^{\rho T} \frac{dV_T}{dT} = v(r_T^T) - r_T^T \, v'(r_T^T) \quad .$$

But then, from (5b), since $\frac{d}{dr}(v(r) - rv'(r)) = -rv''(r) > 0$ for $r > 0$, by (3b),

$$\frac{dV_T}{dT} \begin{bmatrix} < \\ = \\ > \end{bmatrix} 0 \quad \text{for} \quad r_T^T \begin{bmatrix} < \\ = \\ > \end{bmatrix} \hat{r} \ .$$

Finally, since $0 < T < T' < \bar{T}$ implies $r_{T'}^{T'} \leqq r_T^{T'} < r_T^T$,

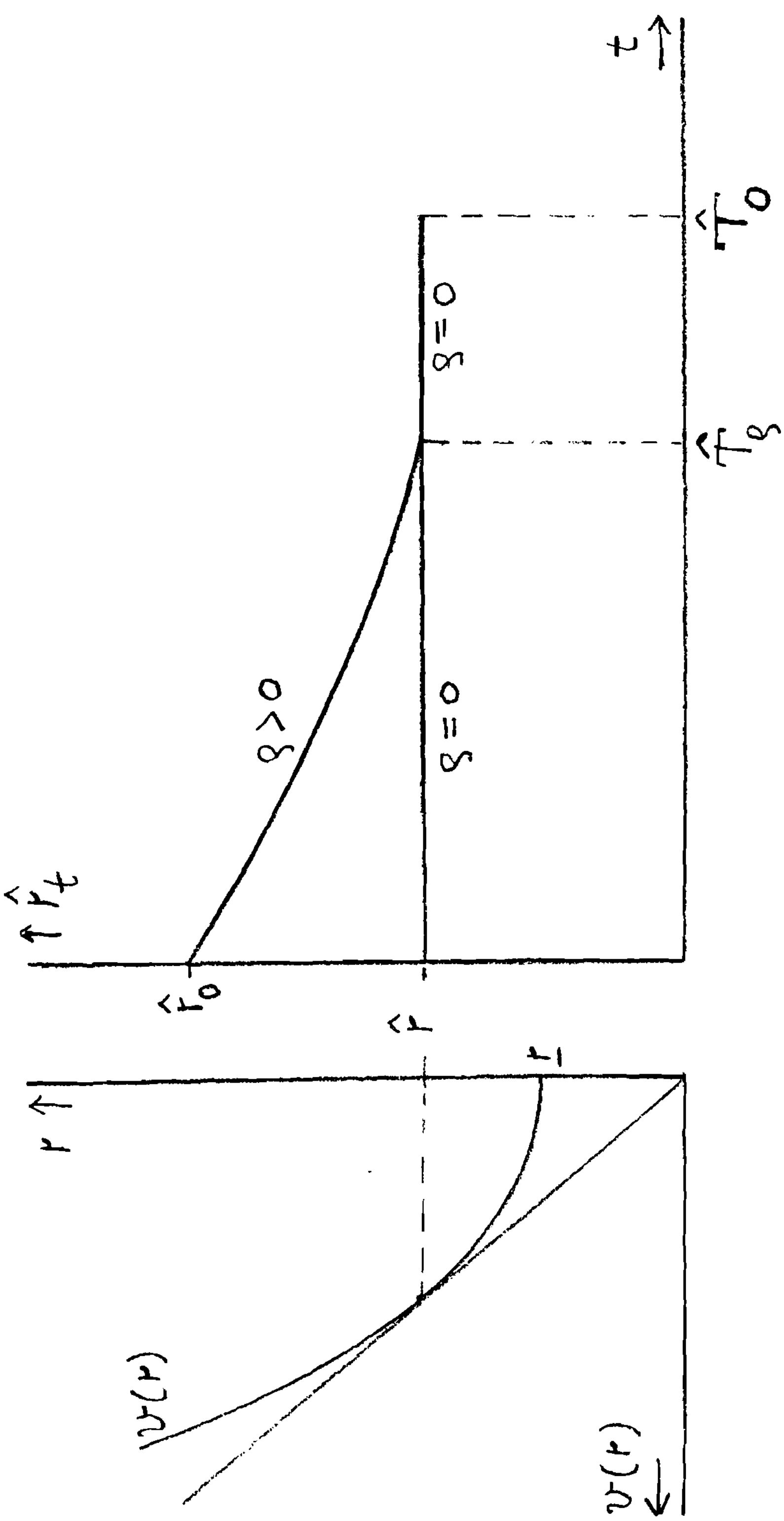
$$\frac{dV_T}{dT} \begin{bmatrix} < \\ = \\ > \end{bmatrix} 0 \quad \text{for} \quad T \begin{bmatrix} > \\ = \\ < \end{bmatrix} \hat{T}_\rho \ .$$

Thus, V_T reaches its unique maximum for that value $\hat{T}_\rho$ of T for which $r_T^T = \hat{r}$.

This establishes the second part of the theorem. The first part follows by specialization when $\rho = 0$.

REFERENCE

Koopmans, T.C., "Some observations on 'optimal' economic growth and exhaustible resources", in Bos, Linnemann and de Wolff, Ed$^{\underline{s}}$, Economic Structure and Development, essays in honour of Jan Tinbergen, Holland Publishing Co., 1973, pp. 239-55.


$\hat{r}_t$
t
$\delta > 0$
$\delta = 0$
$\delta = 0$
$\hat{T}_\delta$
$\hat{T}_0$
$\hat{r}_0$
$\hat{r}$
r_1
r
$v(r)$
$v(r)$